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ON ESSÉN'S GENERALIZATION OF THE AHLFORS-HEINS THEOREM

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ABSTRACT. Recently, Essén has proven a generalization of the Ahlfors-Heins Theorem. In this paper we use Essén's Theorem to obtain a different generalization of the Ahlfors-Heins Theorem.

1. **Introduction.** Let $K = \mathbf{C} - (-\infty, 0]$. Let h be subharmonic in K and put $m(r) = \inf_{|z|=r} h(z)$, $M(r) = \sup_{|z|=r} h(z)$, when $0 < r < \infty$. We also let $\vec{h}(-r) = \overline{\lim}_{z \to -r} h(z)$, when $0 < r < \infty$. Using this notation, Essén's [3] generalization of the Ahlfors-Heins Theorem may be stated as follows:

Theorem A. *Let σ be a number in the interval $(0, 1)$ and let h $(\not\equiv -\infty)$ be a function subharmonic in K that satisfies*

(i) $$\vec{h}(-r) \le \cos \pi\sigma\, h(r), \qquad 0 < r < \infty.$$

Then either $\lim_{r\to\infty} r^{-\sigma}M(r) = \infty$ or both (a) *and* (b) *hold*:

(a) *There exists a number β such that*

(ii) $$\lim_{r\to\infty} r^{-\sigma} h(re^{i\theta}) = \beta \cos \sigma\theta, \qquad |\theta| < \pi,$$

except when $e^{i\theta}$ belongs to a set of capacity zero.

(b) *Given θ_0, $0 < \theta_0 < \pi$, there exists an r-set Δ_0 of finite logarithmic measure such that* (ii) *holds uniformly in $[z: |\theta| \le \theta_0]$ when r is restricted to lie outside of Δ_0.*

Let h be subharmonic in K and $0 < \sigma < 1$. Then if $h(re^{i\theta})/r^{\sigma}$ possesses a limit in the sense of (a) and (b), we shall denote this limit by "lim"$_{r\to\infty} h(re^{i\theta})/r^{\sigma}$.

The author [7] has considered a condition of type (i) in certain regions Ω. In fact for $\Omega = K$, he proved

Theorem B. *Let $0 < \sigma < 1$. Let h be subharmonic in K and suppose that*

(iii) $$\vec{h}(-r) \le \cos \pi\sigma\, M(r)^{+}, \qquad 0 < r < \infty,$$

(iv) $$\vec{h}(-r) < \infty, \qquad 0 \le r < \infty.$$

Then either $h \le 0$ or $\lim_{r\to\infty} M(r)/r^{\sigma}$ exists as a strictly positive or infinite limit.

In this paper we shall use Theorem A and Theorem B to prove the following theorem.

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Theorem 1. *Let* $0 < \lambda < 1$. *Let* u *be subharmonic in* K *and satisfy* (iii) *and* (iv) *with* $h = u$ *and* $\sigma = \lambda$. *If*

$$u(z_0) > 0 \tag{v}$$

for some $z_0 \in K$ *and*

$$\underline{\lim}_{r \to \infty} \frac{M(r)}{r^{\lambda}} = \alpha < \infty, \tag{vi}$$

then "lim"$_{r \to \infty}$ $u(re^{i\theta})/r^{\lambda} = \alpha \cos \lambda\theta$. *Here* $0 < \alpha < \infty$.

We remark that there is an essential difference in Theorem 1 between the two cases $0 < \lambda < \frac{1}{2}$ and $\frac{1}{2} \leq \lambda < 1$. Indeed, suppose that u satisfies the hypotheses of Theorem 1 except for (v). Then if $\frac{1}{2} \leq \lambda < 1$ it follows from the proof of Theorem 1 that "lim" $u(re^{i\theta})/r^{\lambda} = \alpha \cos \lambda\theta$, $|\theta| < \pi$, although in this case α may be zero. However, if $0 < \lambda < \frac{1}{2}$, then "lim"$_{r \to \infty}$ $u(re^{i\theta})/r^{\lambda}$ need not exist (see Lewis [7, §7]).

We shall also derive Theorem A for $0 < \lambda \leq \frac{1}{2}$ from Theorem 1 (see §8).

2. **Proof of Theorem 1 for** $\frac{1}{2} \leq \lambda < 1$. Let $\frac{1}{2} \leq \lambda < 1$ and suppose that u satisfies the hypotheses of Theorem 1. In this case we note that

$$\vec{u}(-r) \leq \cos \pi\lambda\, M(r) \leq \cos \pi\lambda\, u(r)$$

for $0 < r < \infty$, since $\cos \pi\lambda \leq 0$. Therefore we may apply Essén's Theorem. By this theorem we have "lim"$_{r \to \infty}$ $u(re^{i\theta})/r^{\lambda} = \beta \cos \lambda\theta$, where $-\infty < \beta < \infty$. We shall show in fact that

$$\beta = \alpha = \underline{\lim}_{r \to \infty} \frac{M(r)}{r^{\lambda}}. \tag{2.1}$$

Since by (vi) and Theorem B, $0 < \alpha < \infty$, the proof of Theorem 1 for $\frac{1}{2} \leq \lambda < 1$ will then be complete. We shall want the following lemma.

Lemma 1. *Let* h *and* σ *be as in Theorem* A. *If* $\underline{\lim}_{r \to \infty} M(r, h)/r^{\sigma} < \infty$, *then in* (a) *we have* $\beta = \overline{\lim}_{r \to \infty} h(r)/r^{\sigma}$.

Proof. (See Essén [3, Lemma 6.1].)

We now turn to the proof of (2.1). The proof is by contradiction. Let $\beta^+ = \max[\beta, 0]$ and suppose that $\beta^+ < \alpha$. In this case choose $\epsilon > 0$ small enough such that $\beta^+ + 2\epsilon \leq \alpha$. Then by Lemma 1 and Theorem B we see that for R_0 large enough

$$u(r) \leq (\beta^+ + \epsilon) r^{\lambda} \leq (\alpha - \epsilon) r^{\lambda} \leq M(r), \qquad r \geq R_0. \tag{2.2}$$

Consider now the function s defined by

$$s(z) = u(z) - (\beta^+ + \epsilon)\,\mathrm{Re}\,(z^{\lambda}) - C, \qquad z \in K.$$

Here C is a positive constant to be determined, and z^λ denotes the analytic λth power of z in K for which $1^\lambda = 1$. It follows from (iii) and (2.2) that if C is large enough, then $\overline{\lim}_{z\to t}\, s(z) \le 0$, $t \in (-\infty, +\infty)$.

Let $I = \{\operatorname{Im} z > 0\}$ and note that $s|_I$ satisfies the hypotheses of the Phragmén-Lindelöf Theorem (see Heins [5, p. 111]). Then by this theorem and (vi), we have $s|_I \le 0$. Applying the same argument in the lower half plane, we obtain $s \le 0$. This inequality implies that $\alpha \le \beta^+ + \epsilon$. Since $\beta^+ + 2\epsilon \le \alpha$, we have reached a contradiction. Therefore, $0 < \alpha \le \beta^+ = \beta$. Since clearly $\beta \le \alpha$, it follows that $\beta = \alpha$.

3. **Proof of Theorem 1 for** $0 < \lambda < \frac{1}{2}$. Let $0 < \lambda < \frac{1}{2}$. In this case since $\cos \pi\lambda > 0$, we see that (iii) is a weaker restriction on u than (i) of Theorem A. Therefore, we may not apply Essén's Theorem. However, an important part of the proof still follows from Essén's work (see §7).

We shall want the following lemma.

Lemma 2. *Let h and σ be as in Theorem* B. *Then* $M(r, h)^+ = \max\{M(r, h), 0\}$ *is a nondecreasing convex function of* $\log r$ *on* $(0, \infty)$.

Proof. (See Lewis [7, Lemma 1].)

We now begin the proof of Theorem 1 for $0 < \lambda < \frac{1}{2}$. Let $r_0 = |z_0|$ and put $M(r_0) = b$. Then by (v) of Theorem 1 and Lemma 2, we see that $M(r) \ge b > 0$ when $r \ge r_0$. Next, we associate with u a function v which has the following properties:

(3.1) v is subharmonic in $\mathbf{C}$,

(3.2) $m(r, v) = v(-r) = \cos \pi\lambda\, M(r, v)$, $r \ge r_0$,

(3.3) $M(r, v) = M(r, u) - b$, $r \ge r_0$.

Let $u^+ = \max\{u, 0\}$. Then for v we propose the function defined by

$$\begin{aligned} v(z) &= \max\{u^+(z) - b,\ \cos \pi\lambda\,(M(|z|, u^+) - b)\}, \qquad z \in K,\\ v(-r) &= \cos \pi\lambda\,(M(r, u^+) - b), \qquad 0 < r < \infty,\\ v(0) &= \overline{\lim_{z\to 0}}\ v(z). \end{aligned}$$

Clearly v satisfies (3.2) and (3.3). To prove (3.1) we observe from Lemma 2 that

(*) $M(|z|, u^+)$ is continuous and subharmonic in $\mathbf{C} - \{0\}$.

We also observe for $0 < r < \infty$ that

$$\begin{aligned} \vec{u}^+(-r) - b &\le \cos \pi\lambda\, M(r, u^+) - b \\ &= \cos \pi\lambda\,(M(r, u^+) - b) + (\cos \pi\lambda - 1)b < \cos \pi\lambda\,(M(r, u^+) - b), \end{aligned} \tag{3.4}$$

thanks to (iii) and the fact that $\cos \pi\lambda > 0$.

Using the above inequality and (*), we deduce, if $s > 0$ and small, that

$$v(z) = \cos \pi\lambda (M(|z|, u^+) - b), \qquad |z + r| < s.$$

This equality and (*) imply that v is subharmonic at $-r$ when $0 < r < \infty$. Also, (*) implies that v is subharmonic in K. Hence v is subharmonic in $\mathbf{C} - \{0\}$.

It remains to prove that v is subharmonic at zero. For this purpose let $0 < r < \infty$ and suppose that g is the least harmonic majorant of v restricted to $\{|z| < r\} - \{0\}$. Then since v is bounded above in $\{|z| < r\}$, we have

$$v(0) = \overline{\lim_{z \to 0}}\, v(z) \le \overline{\lim_{z \to 0}}\, g(z) = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta})\, d\theta.$$

Hence v has the mean-value property at zero. Since v is clearly upper semicontinuous at zero, we conclude that v is subharmonic at zero and thereupon that (3.1) is true. Actually v is continuous at zero. Indeed, if f is the least harmonic majorant of v in $K \cap \{|z| < 1\}$ and if $c = \cos \pi\lambda \lim_{r \to 0} (M(r, u^+) - b)$, then

$$c \le \underline{\lim_{z \to 0}}\, v(z) \le v(0) \le \overline{\lim_{z \to 0}}\, f(z) = c.$$

We shall use this remark in §4.

4. **A harmonic majorant of $v|_K$.** We now introduce the function w defined by

$$(4.1)\qquad \begin{aligned} w(z) &= \frac{1}{\pi}\int_0^\infty \frac{v(-s)}{\sqrt{s}} \operatorname{Re}\left(\frac{\sqrt{z}}{z+s}\right) ds, \qquad z \in K, \\ w(-r) &= v(-r), \qquad 0 \le r < \infty. \end{aligned}$$

We claim that

(4.2) w is harmonic in K and continuous in $\mathbf{C}$,

(4.3) w is subharmonic in $\mathbf{C}$,

(4.4) $v \le w$,

(4.5) $w(re^{i\theta}) \sim \alpha r^\lambda \cos \lambda\theta$ $(r \to \infty)$ uniformly for $|\theta| \le \pi$.

Claim (4.2) is a direct consequence of the fact that $v(-r) = O(r^\lambda)$ as $r \to \infty$ and the Poisson integral formula for K (see Boas [2, Theorem 6.5.4] for an analogous formula in a half plane).

To prove (4.4), we apply the Phragmén-Lindelöf Theorem to $(v - w)|_K$. Since $M(r, v - w) = O(r^\lambda)$ as $r \to \infty$, we obtain that $v \le w$.

To prove (4.3), we observe that $v = w$ on $(-\infty, 0]$. Since v is subharmonic in $\mathbf{C}$ and $v \le w$, it follows upon examining circumferential means that w is subharmonic at each point of $(-\infty, 0]$. Since w is also harmonic in K, we conclude that (4.3) is true.

(4.5) follows easily from Theorem B and a standard Phragmén-Lindelöf argument. We omit the proof.

5. **Subharmonic functions in C.** We now require a representation formula for a function h subharmonic in $\mathbf{C}$, harmonic at 0, and of order < 1. Such a function may be represented as (see Heins [6])

$$(5.1)\qquad h(z) = h(0) + \int_{|\zeta| < \infty} \log\left|1 - \frac{z}{\zeta}\right| d\mu(\zeta), \qquad z \in \mathbf{C}.$$

Here μ is the Riesz mass associated with h. Let

$$h^*(z) = h(0) + \int_{|\zeta|<\infty} \log \left|1 + \frac{z}{|\zeta|}\right| d\mu(\zeta), \qquad z \in \mathbf{C}. \tag{5.2}$$

It is easily verified that

$$h^*(-r) \le m(r, h) \le M(r, h) \le h^*(r), \qquad 0 < r < \infty, \tag{5.3}$$

$$h^*(-r) + h^*(r) \le h(z) + h(-z), \qquad |z| = r. \tag{5.4}$$

Using (5.3) and (5.4), we obtain for $0 < \sigma < 1$ and $0 < r < \infty$ that

$$\begin{aligned} h^*(-r) - \cos \pi\sigma\, h^*(r) &= h^*(-r) + h^*(r) - (1 + \cos \pi\sigma) h^*(r) \\ &\le h(-r) + h(r) - (1 + \cos \pi\sigma) h(r) = h(-r) - \cos \pi\sigma\, h(r). \end{aligned} \tag{5.5}$$

In the sequel, we use results of Anderson [1, Theorems 1 and 2] and Essén [4, Theorem 1B]. It is more convenient to use them in the form they have in Essén's paper.

We state Essén's Theorem as follows:

Theorem C. *Let h be subharmonic in* $\mathbf{C}$ *and put*

$$I(R, h) = \int_1^R r^{-1-\sigma}[m(r, h) - \cos \pi\sigma\, M(r, h)]\, dr,$$

where $0 < \sigma < 1$ and $1 < R < \infty$. If $R^{-\sigma} M(R, h)$ and $I(R, h)$ have an upper bound, then $I(R, h)$ is bounded below, and $\lim_{R\to\infty} R^{-\sigma} M(R, h)$ exists if and only if $\lim_{R\to\infty} I(R, h)$ exists. If these limits exist, then $\lim_{R\to\infty} R^{-\sigma} M(R, h^*)$ and $\lim_{R\to\infty} I(R, h^*)$ also exist.

6. A lemma. We shall want the following lemma.

Lemma 3. *Let v and w be as in §§3 and 4. Then* $\int_1^\infty [w(r) - v(r)] dr/r^{1+\lambda} < +\infty$.

Proof. We prove Lemma 3 by showing

$$\overline{\lim_{R\to\infty}} \int_1^R [w(r) - M(r, v)] \frac{dr}{r^{1+\lambda}} < +\infty, \tag{6.1}$$

$$\int_1^\infty [M(r, v) - v(r)] \frac{dr}{r^{1+\lambda}} < +\infty. \tag{6.2}$$

To prove (6.1) we first observe from (3.2), (4.1), and (4.4) that

$$m(r, w) = v(-r) = \cos \pi\lambda\, M(r, v) \le \cos \pi\lambda\, M(r, w), \qquad r \ge r_0. \tag{6.3}$$

Moreover, since $\cos \pi\lambda > 0$ we have

$$\begin{aligned} m(r, w) - \cos \pi\lambda\, M(r, w) &\le w(-r) - \cos \pi\lambda\, w(r) \\ &= \cos \pi\lambda (M(r, v) - w(r)), \qquad r \ge r_0. \end{aligned} \tag{6.4}$$

Since $\lim_{r\to\infty} r^{-\lambda} M(r, w) = \alpha$ and (6.3) is true, it follows from Theorem C that $\int_1^\infty [m(r, w) - \cos \pi\lambda\, M(r, w)]\, dr/r^{1+\lambda}$ converges. It now follows from (6.4) that (6.1.) is valid.

To prove (6.2) we let $v = h$ in (5.1) and $v^* = h^*$ in (5.2). Then by (5.3), (5.4), and (3.2) we have, for $r \geq r_0$,

$$v^*(-r) - \cos \pi\lambda\, v^*(r) = v^*(-r) + v^*(r) - (1 + \cos \pi\lambda) v^*(r)$$

$$\leq v(-r) + v(r) - (1 + \cos \pi\lambda) M(r, v) = v(r) - M(r, v).$$

Using this inequality and Theorem C it follows that (6.2) is true.

7. **The final proof.** Let

$$p(z) = \min\{\operatorname{Re}(z^{\frac{1}{2}}),\ w(z) - v(z)\}, \qquad z \in K.$$

Then p is a nonnegative superharmonic function in K and, by Lemma 3,

$$\int_0^\infty p(r) \frac{dr}{r^{1+\lambda}} \leq \int_0^1 r^{-\frac{1}{2}-\lambda}\, dr + \int_1^\infty [w(r) - v(r)] \frac{dr}{r^{1+\lambda}} < \infty. \tag{7.1}$$

We shall use (7.1) to prove that

$$\int_0^\infty [p(it) + p(-it)] \frac{dt}{t^{1+\lambda}} < \infty. \tag{7.2}$$

Using the Poisson integral formula for a half plane, we find that

$$p(r) \geq \frac{r}{\pi} \int_0^\infty \frac{[p(it) + p(-it)]}{t^2 + r^2}\, dt.$$

Hence,

$$\infty > \int_0^\infty p(r) \frac{dr}{r^{1+\lambda}} > \int_0^\infty p(it) + p(-it) \left(\frac{1}{\pi} \int_0^\infty \frac{r^{-\lambda}}{t^2 + r^2}\, dr \right) dt.$$

$$= \frac{1}{\cos \pi\lambda} \int_0^\infty [p(it) + p(-it)] \frac{dt}{t^{1+\lambda}}.$$

(7.1) and (7.2) imply that

$$\text{``}\lim_{r\to\infty}\text{''} \frac{p(re^{i\theta})}{r^\lambda} = 0. \tag{7.3}$$

The proof is given in Essén [3, §9]. We omit the details.

From (7.3) we see that "lim"$_{r\to\infty}$ $(w(re^{i\theta}) - v(re^{i\theta}))/r^\lambda = 0$. In view of (4.5), it follows that "lim"$_{r\to\infty}$ $v(re^{i\theta})/r^\lambda = \alpha \cos \lambda\theta$. Using this equality and Theorem B, we now deduce that "lim"$_{r\to\infty}$ $u(re^{i\theta})/r^\lambda = \alpha \cos \lambda\theta$. This completes the proof of Theorem 1.

8. **Remark.** Essén's Theorem is easily derived from Theorem 1 for $0 < \lambda \leq \frac{1}{2}$. Indeed, if u satisfies the hypotheses of Theorem A with $u = h$ and $\lambda = \sigma$, then for $0 < r < \infty$, $\vec{u}(-r) \leq \cos \pi\lambda\, u(r) < \cos \pi\lambda\, M(r)$, and $\vec{u}(-r) \leq \cos \pi\lambda\, u(r) < \infty$. Moreover, it is an unessential restriction to assume that $\vec{u}(0) < \infty$ (see Essén [3, (3.1)]). Consider now the function u_1 defined by

$$u_1(z) = u(z) + C \operatorname{Re}(z^\lambda), \qquad z \in K.$$

Here C is chosen large enough such that $u_1(z_0) > 0$ for some $z_0 \in K$.

From our previous remarks we see that u_1 satisfies the hypotheses of Theorem 1, except possibly for (vi). Hence either

$$\infty = \lim_{r \to \infty} \frac{M(r, u_1)}{r^\lambda} = \lim_{r \to \infty} \frac{M(r, u)}{r^\lambda}$$

or

$$\text{``}\lim_{r \to \infty}\text{''} \frac{u_1(re^{i\theta})}{r^\lambda} = \text{``}\lim_{r \to \infty}\text{''} \frac{u(re^{i\theta})}{r^\lambda} + C \cos \lambda\theta.$$

In either case we obtain Essén's Theorem.

Finally, we remark for $0 < \alpha < \infty$ that $u(z) = -\alpha \operatorname{Re}(z^\lambda)$, $z \in K$, satisfies the hypotheses of Theorem A but not of Theorem 1 for $0 < \lambda < \frac{1}{2}$.

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